Cantor sets with high-dimensional projections: existence and exceptionality

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L. Antoine constructed a Cantor set in $\mathbb{R}^2$ whose projections coincide with those of a regular hexagon [2, **9**, p.272; fig.2, p.273]. K. Borsuk described a Cantor set in $\mathbb{R}^n$ such that its projection onto every hyperplane contains an $(n-1)$-dimensional ball, equivalently, has dimension $(n-1)$ [4]. J. Cobb [6] gave an example of a Cantor set in $\mathbb{R}^3$ such that its projection onto every 2-plane is one-dimensional, and posed a general question: "Could there be Cantor sets all of whose projections are connected, or even cells? ...given $n > m > k > 0$, does there exist a Cantor set in $\mathbb{R}^n$ such that each of its projections into m-planes is exactly k-dimensional?" (Such sets are called (n, m, k)-sets for brevity.) In case of $m = k$, a positive answer is given by [4]. For cases $(n, m, k = m-1)$ and $(n, m = n-1, k)$, such sets were constructed in [6] and [3], correspondingly; both papers extend Cobb's ideas.

Cobb's method is rather sophisticated; the resulting Cantor set (and also the sets from [3], [6]) is tame in the following sense.

Definition. A zero-dimensional compact set $K \subset \mathbb{R}^n$ is called *tame* if there exists a homeomorphism h of $\mathbb{R}^n$ onto itself such that $h(K)$ is a subset of a straight line in $\mathbb{R}^n$; and it is called *wild* otherwise.

In $\mathbb{R}^2$ each zero-dimensional compactum is tame [1, **75**, p. 87–89]. L. Antoine constructed a family of Cantor sets in $\mathbb{R}^3$ which are now widely known as Antoine's necklaces [1, **78**, p. 91–92] and proved that they are wild [1, Part 2, Chap. III].

Applying the theory of tame and wild Cantor sets, we get new examples of $(n, n-1, n-1)$- and $(n, n-1, n-2)$-sets. It turns out that examples can be obtained from any given Cantor set. For case $(3, 2, 1)$, we get even simpler examples. Finally, we show that Cantor sets which have a high-dimensional projection are, in a sense, exceptional. Our results are the following (for details, see [7], [8], [9]):

✓ Let $K \subset \mathbb{R}^n$ be any Cantor set, $n \geqslant 2$. For each $\varepsilon > 0$ there exists an ε-isotopy $\{h_t\} : \mathbb{R}^n \cong \mathbb{R}^n$ such that $h_1(K)$ is an $(n, n-1, n-1)$-set.

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✓ For the case $(3, 2, 1)$, we present another very simple series of Cantor sets in $\mathbb{R}^3$ all of whose projections are connected and one-dimensional. These are self-similar Antoine's necklaces which satisfy some additional conditions; self-similarity omits the necessity of additional isotopical transformations.

✓ J. Cobb remarked that Cantor sets that raise dimension under all projections and those that do not are both dense in the Cantor sets in $\mathbb{R}^n$ [5, p. 128]. We show that all projections of a typical (in the sense of Baire category) Cantor set are Cantor sets, partially answering another question of Cobb.

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References

[1] L. Antoine. Sur l'homéomorphie de deux figures et de leurs voisinages. — Thèses de l'entre-deux-guerres, vol. 28, 1921. http://eudml.org/doc/192716

[2] L. Antoine. Sur les voisinages de deux figures homéomorphes // Fund. Math. 5 (1924) 265–287.

[3] S. Barov, J.J. Dijkstra, M. van der Meer. On Cantor sets with shadows of prescribed dimension // Topol. Appl. 159 (2012) 2736–2742.

[4] K. Borsuk. An example of a simple arc in space whose projection in every plane has interior points // Fund. Math. 34 (1947) 272–277.

[5] J. Cobb. Raising dimension under *all* projections // Fund. Math. 144 (1994) 2, 119–128.

[6] O. Frolkina. A Cantor set in with “large” projections // Topol. Appl., 157 (2010) 4, 745–751.

[7] O. Frolkina. Cantor sets with high-dimensional projections // Topol. Appl. 275 (2020) 107020.

[8] O. Frolkina. All projections of a typical Cantor set are Cantor sets // Topol. Appl. (2020) 107192.

[9] O. Frolkina. A new simple family of Cantor sets in $\mathbb{R}^3$ all of whose projections are one-dimensional // Topol. Appl.